[1] "Imprimitivity for Representations of Locally Compact Groups I," PROC. NATL. ACAD. SCI., **35**, 537–545 (1949).

[2] Peter, F., and Weyl, H., "Die Vollständigkeit der primitiven Darstellungen einer geschlossenen kontinuierlichen Gruppe," *Math. Ann.* **97**, 737–755 (1927).

[3] Cf. Mautner, F. I., "Unitary Representations of Locally Compact Groups I," *Ann. Math.*, **51**, 1–25 (1950), especially theorem 1.2, and "Unitary Representations of Locally Compact Groups II," *Ibid.*, **52**, 528–556 (1951), especially theorem 1.1. These two papers will be cited as UR I and UR II, respectively.

[4] G. W. Mackey has decomposed the regular representation of certain discrete groups explicitly such that each irreducible component representation occurs with multiplicity one (oral communication of an unpublished result).

[5] Cf. von Neumann, J., "On Rings of Operators, Reduction Theory," *Ann. Math.*, **50**, 401–485 (1949), especially theorems IV and VII.

[6] See theorem 1.1 of UR II.

[7] If G is a connected Lie group then the stronger result holds that N_t can be taken to be empty. This is proved in "On the Decomposition of Unitary Representations of Lie Groups" which is to appear in *Proc. Am. Math. Soc.*

[8] Mackey, G. W., "Imprimitivite pour les representations des groupes localement compact II," *C. R. Ac. Sci. Paris*, **230**, 808–809 (1950), and "III," pp. 908–909.

[9] See Gelfand, "Spherical Functions in Symmetric Riemann Spaces," *Dokladi Acad. Nauk*, **70**, 5–8 (1950). Cf. also Harish-Chandra, "Representations of Semisimple Lie Groups on a Banach Space," PROC. NATL. ACAD. SCI., **37**, 170–173 (1951).

HEAT CONDUCTION ON RIEMANNIAN MANIFOLDS II: HEAT DISTRIBUTION ON COMPLEXES AND APPROXIMATION THEORY

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In a preceding note,[1] hereafter denoted by H. C. I, the equation (1) $\Delta\alpha = \partial\alpha/\partial t$ was discussed, for α an exterior differential form defined on a closed orientable Riemannian manifold V_n of class C^r, $r \geq 5$, where the coefficients of α are functions of a parameter t. We described the method of construction of a double p-form $J_p(x, y, t)$ called the fundamental solution of (1). The form $J_p(x, y, t)$ is the analog for forms of the temperature distribution at time t obtained by placing an infinite source with unit energy at the point y when $t = 0$. It has the following properties:

(1) If $T_t\alpha_0(x) = \int_{V_n} J_p(x, y, t)\alpha_0^*(y)$, then $T_t\alpha_0$ is the solution at time t of (1) such that $\lim_{t\to 0} T_t\alpha_0 = \alpha_0$. This initial condition uniquely determines the solution of (1).

(2) J_p is of class C^{r-2} for $t > 0$.

(3) $J_p(x, y, s + t) = \int_{V_n} J_p(x, z, s)J_p^*(z, y, t)$ where the integration and $*$ operation are performed with respect to the variable z.

(4) $J_p(x, y, t)$ is a solution of (1) for each y and $t > 0$.

(5) $J_p(x, y, t) = J_p(y, x, t)$.

(6) $d_x J_p(x, y, t) = -\delta_y J_{p+1}(x, y, t)$.

(7) $J_p - w_p$ approaches zero uniformly as $t \to 0$, where w_p is the parametrix defined in H. C. I.

Properties (1) and (2) were proved in H. C. I. Property (3) follows from (1) and the semigroup property $T_{s+t} = T_s T_t$. This now implies property (4). Property (5) follows from the equation $(T_t\alpha, \beta) = (\alpha, T_t\beta)$ proved previously. Property (6) follows from (1) and the equations $dT_t = T_t d$, $\delta T_t = T_t \delta$. Finally (7) follows from the integral equation satisfied by $J_p' = J_p - w_p$, namely,

$$J_p'(x, y, t) = \int_0^t (L_1 w_p(x, z, t - \tau), w_p(z, y, \tau))\, d\tau + \int_0^t (L_1 w_p(x, z, t - \tau), J_p'(z, y, \tau))\, d\tau,$$

where the integrations and differentiations are with respect to z and τ.

To each smooth singular p-chain C^p we associate a form

$$T_t C^p = \int_{C^p} J_p(x, y, t),$$

where the integration is with respect to y. The associated form $T_t C^p$ is a solution of (1) analogous to the temperature distribution at the time t resulting from a distribution of heat sources on C^p and zero temperature on the rest of the manifold at time $t = 0$. This mapping of chains into forms has the following properties:

(*a*) $(\alpha, T_t C^p) = \int_{C^p} T_t \alpha$.

(*b*) $T_t \partial C^p = -\delta T_t C^p$.

(*c*) $T_t C^p \equiv 0$ for a fixed $t > 0$ if and only if $C^p = 0$. We regard two chains C_1^p and C_2^p as equal if $\int_{C_1^p} \alpha = \int_{C_2^p} \alpha$ for all p-forms α of class C'. For chains in a given non-degenerate complex in V_n equality is the same as identity.

(*d*) For any fixed $t > 0$ and any fixed sequence of subdivisions of a given triangulation of V_n such that the mesh approaches zero, the forms $T_t C^p$, where C^p ranges over all finite chains formed from simplexes in these subdivisions, are dense in the Hilbert space H of all forms α whose coefficients are measurable and such that $(\alpha \cdot \alpha^*)^*$ is integrable over V_n.

If K is any triangulation of V_n, and

$$C_i^p = \Sigma_j a_{ij} \sigma_j^p, \qquad i = 1, 2,$$

where the σ_j^p are the p-simplices of K, the scalar product of chains[2] is usually defined by the formula

$$C_1^p \cdot C_2^p = \Sigma_j a_1^j a_2^j.$$

Actually, for most purposes, any definition of the form

$$C_1^p \cdot C_2^p = \sum_{j,\, k} g_{jk} a_1^j a_2^k$$

would serve equally well, provided only that the quadratic form $g_{jk}u^j u^k$ be positive definite. Of course, the coboundary operator in this case should be taken as the adjoint of the boundary operator. For our purposes it is convenient to define the scalar product of chains by

$$C_1^p \cdot C_2^p = (T_{t_0} C_1^p,\ T_{t_0} C_2^p), \tag{2}$$

so that

$$g_{jk} = (T_{t_0}\, \sigma_j^p,\ T_{t_0} \sigma_k^p).$$

This has the advantage that the scalar product of chains is invariant under subdivision of K, and has therefore a direct geometric meaning. The corresponding norm, $||C^p|| = +\sqrt{C^p \cdot C^p}$, measures not only the sizes of the coefficients of C^p but also, in a sense, its p-dimensional volume. We note that in (2) the time $t_0 > 0$ is fixed and can be chosen equal to 1 for normalization purposes. If C^p is different from zero in the sense of (c) then $||C^p|| > 0$.

It follows from (b) that if Z^p is a cycle, then $\delta T_t Z^p = 0$, i.e., T_t maps cycles into coclosed forms, and similarly it maps bounding cycles into coexact forms.

Let $K_1, K_2, \ldots$ be a sequence of subdivisions of a given triangulation of V_n with mesh approaching zero. Let $B_1^p, B_2^p \ldots$ form an orthonormal basis for the linear space of bounding cycles from all K_n of the subdivisions $\{K_n\}$.

If α is any form in H, its Fourier expansion $P(\alpha) = \Sigma c_\nu T_1 B_1^p$, where $c_\nu = (\alpha, T_1 B_\nu^p)$, converges in the norm of H. Hence $\alpha = P\alpha + \phi$ where ϕ is orthogonal to $T_1 B^p$ for an arbitrary bounding cycle B^p. By a direct application of Stokes' theorem we see that $T_t\phi$ is closed for all $t > 0$. $P\alpha$ is a limit of coexact forms, hence $T_t P\alpha$ is certainly coclosed. Therefore if α is coclosed, $T_t\alpha$ is also and it follows that $T_t\phi$ is also coclosed. Thus $T_t\phi$ is harmonic for all $t > 0$, and consequently ϕ differs from $T_t\phi$ (which is independent of t) only on a set of measure 0, and therefore may be identified, in H, with a harmonic form, and $P\alpha$ may be regarded as smooth. If α is an arbitrary form in H, applying the same procedure to ϕ^* and then starring the result demonstrates that each form α in H is expressible as the sum of a harmonic form and two forms in the closures in H of the exact and coexact forms, respectively.

Noting from the decomposition theorem of DeRham (cf. H. C. I.) that there is a decomposition and that it is unique, we find that $P\alpha$ is coexact if α is continuous, and $(P\phi^*)^*$ is exact. We have thus shown that these Fourier expansions converge in the mean to continuous exact and coexact forms. Thus our procedure bridges the gap between finite complexes and forms. We note that the coefficients c_ν and hence the chains $\Sigma c_\nu B_\nu^p$ are uniquely associated with $P\alpha$.

Let us apply these results to $T_1 Z^p$ where Z^p is a non-bounding cycle. Then $T_1 Z^p = P(Z^p) + \phi$ where ϕ is harmonic, and orthogonal to $P(Z^p)$. We have

$$\int_{Z^p}\phi = \int_{Z^p} T_1\phi = (T_1 Z^p, \phi) = ||\phi||^2,$$

so that the period of ϕ on Z^p will be different from zero if ϕ is not identically zero. Let Z^{n-p} be an arbitrary $n - p$-cycle. $T_t Z^{n-p}$ is a coclosed form and therefore is orthogonal to $T_t P(Z^p)^*$ since the latter is a limit of exact forms. Thus

$$(T_t Z^{n-p}, \phi^*) = (T_t Z^{n-p}, (T_t Z^p)^*) = \int_{Z_y^{n-p}} \int_{Z_x^p} J_p(x, y^*, 2t),$$

where the subscripts x, y denote the variables with respect to which the integrations are performed. Using (7) above and examining the resulting integral as $t \to 0$, we find $\lim\limits_{t \to 0} (T_t Z^{n-p}, \phi^*)$ is equal to the intersection number of Z^{n-p} and Z^p. Since Z^p is a non-bounding cycle, there exists a Z^{n-p} for which this intersection number is different from zero. Hence $\phi \neq 0$.

Since Hodge has shown that a harmonic form whose periods are zero must be identically zero, we have:

THEOREM I. *If* $HC^p = \lim\limits_{t \to \infty} T_t C^p$, *then* HC^p *is always a harmonic form. If* $Z_1^p \ldots, Z_\beta^p$ *is a basis of p-cycles linearly independent with respect to homology in a fixed triangulation of* V_n, *then* $HZ_1^p, \ldots, HZ_\beta^p$ *is a linearly independent basis for all the harmonic p-forms. The period of* HZ_i^p *will be different from zero on the cycle* Z_i^p. *Here* $\beta = \beta^p$ *is the* pth *Betti Number of* V_n.

The notion of a heat distribution on a chain is related to the theory of currents developed by DeRham.[3] T_t is a smoothing operator which applies, in fact, to arbitrary currents, and provides a powerful tool for their investigation. We hope to develop this approach in a later paper.

[1] These PROCEEDINGS, **37**, 180–184 (1951).

[2] Lefschetz, S., *Algebraic Topology* (Am. Math. Society Colloquium Publications), New York, 1942.

[3] DeRham, "Notes on Lectures of Institute for Advanced Study," 1940.